RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2019
SECOND YEAR (BATCH 2017-20)
MATHEMATICS (Honours)
Paper : IV

Date : 16/05/2019
Time : 11.00 am – 3.00 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

***Answer <u>any five</u> questions* from <u>question nos. 1 to 8</u> :** [5×7]

1. a) If (X, d) is a metric space then show that $d: X\times X \to [0,\infty)$ is continuous (X × X is equipped with product metric D defined by $D\big((x_1,y_1),(x_2,y_2)\big)=\max\{d(x_1,x_2),d(y_1,y_2)\}$ for all $(x_1,x_2),(y_1,y_2)\in X\times X$) .

   b) Let (X, d) be a metric space, A be a compact subset of X and $d(x,A)=\operatorname*{Inf}_{a\in A} d(x,a)$.
   Show that for any $x\in X$, there exists $y\in A$ such that $d(x,A)=d(x,y)$. (3+4)

2. a) Let (X, d) be a metric space and $A\subseteq X$. Prove that diam (A) = diam $(\overline{A})$ and also that $\overline{A}=\{x\in X: d(x,A)=0\}$ where $\overline{A}$ denotes the smallest closed set containing A.

   b) If (X, d) is a metric space and $x_0\in X$ then show that $\{x\in X: d(x,x_0)\le 1\}$ is a closed set. [(3+2)+2]

3. a) Show that every metric space is normal.

   b) Suppose $X=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}$. Define metrics d_1 and d_2 as follows:
   $d_1: X\times X\to\mathbb{R}$ by $d_1(x,y)=|x-y|$ & $d_2: X\times X\to\mathbb{R}$ by $d_2(x,y)=\begin{cases}1, & x\neq y\\ 0, & x=y\end{cases}$
   Are the metrics d_1, d_2 equivalent? Justify your answer. (4+3)

4. a) Prove that every countably compact metric space is pseudocompact.

   b) Show that every pseudocompact metric space is complete. (3+4)

5. a) Find an open cover of $\mathbb{Q}\cap[0,1]$ having no finite subcover. Justify your answer.

   b) If X is a compact metric space, show that every open cover of X has a Lebesgue number. (3+4)

6. a) Give an example of a metric space which is of first category.

   b) State and prove Baire's category theorem. (2+5)

7. a) Let A be a closed set in a metric space (X,d). Show that there is a continuous map $f: X\to\mathbb{R}$ such that $\{x\in X: f(x)=0\}=A$. Hence prove that A is G_δ.

   b) Show that $\mathbb{Q}$ is not G_δ in $\mathbb{R}$. (4+3)

8. a) Show that $\mathbb{Q}$ is not connected.

   b) Suppose $f:\mathbb{R}\to\mathbb{R}$ is a bounded continuous map. Show that f has a fixed point. (3+4)

Answer <u>any three</u> questions **from <u>question nos. 9 to 13</u> :** [3×5]

9. Let W be a subspace of an inner product space V and β be a vector in V. Prove that the vector α in W is a best approximation to β by vectors in W if and only if $\beta-\alpha$ is orthogonal to every vector in W. (5)

10. Let T be a normal operator on a Hermitian space V, and let ν be an eigen vector of T with eigen value λ. Then v is also an eigen vector of T^*, with eigen value $\overline{\lambda}$ (here T^* is the adjoint operator of T). (5)

11. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A=\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ (5)

12. Let V be an inner product, and let T be a normal operator on V. Prove that the following statements are true:
    a) $\|Tx\|=\|T^*x\|$ for all $x\in V$.
    b) If λ_1 and λ_2 are distinct eigen values of T with corresponding eigen vectors x_1 and x_2, then x_1 and x_2 are orthogonal. (2+3)

13. Let $V=C([0,1])$ with the inner product $(f\,|\,g)=\int_0^1 f(t)g(t)\,dt$. Let W be the subspace spanned by the linearly independent set $\{t,\sqrt{t}\}$.
    a) Find an orthonormal basis for W.
    b) Use the orthonormal basis obtained in (a) to obtain the best approximation of t^2 in W. (3+2)

<u>Group – B</u>

Answer <u>any three</u> questions **from <u>question nos. 14 to18</u> :** [3×5]

14. Solve the equation $x^2\frac{d^2y}{dx^2}-(x^2+2x)\frac{dy}{dx}+(x+2)y=x^3e^x$ in terms of known integral. (5)

15. Solve the equation $\frac{d^2y}{dx^2}-2\tan x.\frac{dy}{dx}-2y=e^x.\sec x$, by reducing it to normal form. (5)

16. Find the eigen values λ_n and the eigen functions $y_n(x)$ for the differential equation $\frac{d^2y}{dx^2}+\lambda y=0$ satisfying the boundary conditions $y(0)=0$ and $y(\pi)=0$. (5)

17. Solve the system of differential equations $(D^2-2)x-3y=e^{2t}$; $(D^2+2)y+x=0$. Find also the particular solution if the initial conditions are x = y = 1; Dx = Dy = 0 when t = 0 and $D\equiv\frac{d}{dt}$. (5)

18. Solve the equation $\frac{d^2y}{dx^2}+(x-1)^2\frac{dy}{dx}-4(x-1)y=0$ in series about the ordinary point x = 1. (5)

Answer ***<u>any three</u>*** ***questions*** **from <u>question nos. 19 to 23</u> :** [3×5]

19. Solve the equation $\left(2x^2+2xy+2xz^2+1\right)dx+dy+2zdz=0$ after satisfying the condition of integrability. (1+4)

20. Find a complete integral of $2(z+px+qy)=yp^2$ using Charpit's auxiliary equation. (5)

21. a) Evaluate $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\}$

b) Find $L\left\{\sin\sqrt{t}\right\}$ (2+3)

22. a) Use convolution theorem to find $L^{-1}\left\{\frac{p}{\left(p^2+a^2\right)^2}\right\}$.

b) If F(t) be a periodic function with period T > 0, then prove that $L\{F(t)\}=\frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$. (2+3)

23. Using Laplace Transform technique , solve $\left\{tD^2+(1-2t)D-2\right\}y=0$, $D\equiv\frac{d}{dt}$ given $y(0)=1$ and $y'(0)=2$. (5)

Answer ***<u>any two</u>*** ***questions*** **from <u>question nos. 24 to 26</u> :** $[2\times10]$

24. a) Prove that the locus of the extremity of the polar subtangent of the curve $u=f(\theta)$ is $u+f'\left(\frac{\pi}{2}+\theta\right)=0$ ['$'$' represents the differentiation w.r.t. θ].

b) Tangents are drawn from the origin to the curve $y=\sin x$. Show that their points of contact lie on the curve $x^2y^2=x^2-y^2$.

c) Show that the pedal equation of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is $r^2+3p^2=a^2$ (4+3+3)

25. a) Find the evolute of the parabola $y^2=12x$.

b) Prove that the asymptotes of the cubic $x^2y-xy^2+xy+y^2+x-y=0$ cut the curve again in three points which lie on the line x+ y = 0. (4+6)

26. a) Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters are connected by $a^2+b^2=c^2$ (c being a given constant).

b) Show that the curve $y=\sin\frac{x}{a}$ has a point of inflexion whenever the curve crosses the x-axis.

c) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is revolved about the line y = b. Find the volume of the solid thus generated. (4+3+3)

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